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$$x^2 - y^2 + \frac{R\sqrt{21}}{7}x + \frac{13R\sqrt{7}}{21}y = 0$$

with the diameter BG prolonged as the axis of X and AY as the axis of Y. Then will the ordinate of intersection at D minus $R\sqrt{7}/3$ be the side of a regular heptagon inscribed in a circle whose radius is R; that is,

$$z = \frac{R\sqrt{7}}{3}[2 \sin \tfrac{1}{3}(\pi + \sin^{-1} \tfrac{13}{14}) - 1]$$

is the side of the regular heptagon required.

III. On the Summation of Certain Series.

By O. Schmiedel, Parsons College.

The summation of a finite number of terms of series such as

$$1 + 3x + 6x^2 + 10x^3 + \cdots,$$

$$1 + \cos\theta + \cos 2\theta + \cos 3\theta + \cdots,$$

$$1 + x\cos\theta + x^2\cos 2\theta + x^3\cos 3\theta + \cdots,$$

$$1 + 3x\cos\theta + 6x^2\cos 2\theta + 10x^3\cos 3\theta + \cdots,$$

and the evaluation of the complete series is a problem of frequent occurrence, and is easily solved by the use of De Moivre's Theorem;[1] but greater insight to the student into their nature and mutual relation is had by direct application of the more common processes of analysis.

It is proposed to produce a formula sufficiently comprehensive to embrace all series of the types suggested, including as particular cases the binomial formula for negative integral exponents and the theorem that the difference of the same powers is divisible by the difference of the first.

Let the second of the series above be deduced first, for illustration.

Assuming the identity

$$2\sin\theta/2\cos t\theta = \sin(t + 1/2)\theta - \sin(t - 1/2)\theta,$$

assigning to t values from 0 to n, and adding the results, gives directly the expression for the sum of the series of cosines, namely:

$$(1) \qquad \sum_{t=0}^{t=n}\cos t\theta = 1/2 + 1/2\frac{\sin(n + 1/2)\theta}{\sin\theta/2} = \frac{1}{\sin\theta/2}\sin\frac{n+1}{2}\theta\cos\frac{n}{2}\theta.$$

In like manner is found the sum of sines:

$$(2) \qquad \sum_{t=0}^{t=n}\sin t\theta = \frac{1}{\sin\theta/2}\sin\frac{n+1}{2}\theta\sin\frac{n}{2}\theta;$$

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[1] Chrystal, *Algebra*, Vol. II, 2 ed., p. 273.

and the quotient of (2) by (1) gives the interesting relation

$$\frac{\sum_{t=0}^{t=n} \sin t\theta}{\sum_{t=0}^{t=n} \cos t\theta} = \tan \frac{n}{2}\theta,^{1} \tag{3}$$

an extension of the familiar formula $\tan \theta/2 = \sin \theta/(1 + \cos \theta)$.

Formula (1) may also be written thus:

$$\sum_{t=0}^{t=n} \cos t\theta = 1/2 + \frac{\cos n\theta - \cos(n+1)\theta}{2(1 - \cos\theta)}, \tag{4}$$

$$-\sum_{t=-n+1}^{t=-1} \cos t\theta = 1/2 + \frac{\cos n\theta - \cos(n-1)\theta}{2(1 - \cos\theta)}. \tag{5}$$

Analogously the summation of the other series is effected, as:

$$\sum_{t=0}^{t=n} x^t \cos t\theta = \frac{1 - x\cos\theta - x^{n+1}\cos(n+1)\theta + x^{n+2}\cos n\theta}{1 - 2x\cos\theta + x^2}, \tag{6}$$

$$-\sum_{t=-n+1}^{t=-1} x^t \cos t\theta = \frac{1 - x\cos\theta - x^{-n+1}\cos(n-1)\theta + x^{-n+2}\cos n\theta}{1 - 2x\cos\theta + x^2}, \tag{7}$$

one or the other of which is convergent for a given $x \neq \pm 1$, as n becomes infinite. Using the following definition and identities:

$$c_{a,\,b} = \frac{a(a-1)\cdots(a+1-b)}{1.\ 2.\ \cdots b};$$

$$c_{a,\,b} + c_{a,\,b+1} = c_{a+1,\,b+1}, \qquad c_{a,\,b} = \sum_{t=0}^{t=n} (-1)^t c_{n,\,t}c_{a+n-t,\,b-t};$$

$$2\cos u\cos v = \cos(u+v) + \cos(u-v),$$

$$\cos r\theta(1 - 2x\cos\theta + x^2)^m = \sum_{t_1=0}^{t_1=2m}\sum_{t=0}^{t=t_1} c_{m,\,t_1-t}c_{m,\,t}(-x)^{t_1}\cos(r + 2t - t_1)\theta;$$

the sums to n terms are found successively by the same process for all the series under this head, as:

$$\begin{aligned}(1 - 2x\cos\theta + x^2)^2 \sum_{t=0}^{t=n} c_{1+t,}\, x^t \cos t\theta &= \sum_{t=0}^{t=2} (-1)^t c_{2,\,t}x^t\cos t\theta \\ &+ \sum_{t_2=0}^{t_2=3}\sum_{t_1=0}^{t_1=1}\sum_{t=0}^{t=1} (-1)^{t_2+t}c_{2,\,t_2-t_1}c_{2,\,2+t_1-t}c_{n+t,\,1}x^{n+1+t_2}\cos(n+1+2t_1-t_2)\theta,\end{aligned} \tag{8}$$

$$\begin{aligned}(1 - 2x\cos\theta + x^2)^3 \sum_{t=0}^{t=n} c_{2+t,\,2}x^t \cos t\theta &= \sum_{t=0}^{t=3} (-1)^t c_{3,\,t}x^t\cos t\theta \\ &- \sum_{t_2=0}^{t_2=5}\sum_{t_1=0}^{t_1=2}\sum_{t=0}^{t=2} (-1)^{t_2+t}c_{3,\,t_2-t_1}c_{3,\,3+t_1-t}c_{n+t,\,2}x^{n+1+t_2}\cos(n+1+2t_1-t_2)\theta,\end{aligned} \tag{9}$$

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[1] *Ibid.*, p. 327 ff.

etc., and finally the general formula sought:

$$
\begin{aligned}
(10)\qquad (1 - 2x\cos\theta + x^2)^m \sum_{t=0}^{t=n} c_{m-1+t,\, m-1} x^t \cos t\theta &= \sum_{t=0}^{t=m} (-1)^t c_{m,\, t} x^t \cos t\theta \\
&+ (-1)^m \sum_{t_2=0}^{t_2=2m-1} \sum_{t_1=0}^{t_1=m-1} \sum_{t=0}^{t=m-1} (-1)^{t_2-t} c_{m,\, t_2-t_1} c_{m,\, m+t_1-t} c_{n+t,\, m-1} \\
&\times x^{n+1+t_2} \cos(n+1+2t_1-t_2)\theta,
\end{aligned}
$$

$$
\begin{aligned}
(11)\qquad -(1 - 2x\cos\theta + x^2)^m \sum_{t=-n+1}^{t=-1} c_{m-1+t,\, m-1} x^t \cos t\theta &= \sum_{t=0}^{t=m} (-1)^t c_{m,\, t} x^t \cos t\theta \\
&+ (-1)^m \sum_{t_2=0}^{t_2=2m-1} \sum_{t_1=0}^{t_1=m-1} \sum_{t=0}^{t=m-1} (-1)^{t_2-t} c_{m,\, t_2-t_1} c_{m,\, m+t_1-t} c_{-n+t,\, m-1} \\
&\times x^{-n+1+t_2} \cos(-n+1+2t_1-t_2)\theta,
\end{aligned}
$$

where n and m are positive.

Of the two expressions in the second member of (10), which constitute the sum of the series in the first, one is free from n, hence constant as to n; the other contains n, and hence changes with it. If for increasing n the second becomes less and less, with zero as limit, the series is convergent.

A rigorous proof of (10) by the $(n+1)$-rule, though somewhat lengthy, is not difficult to give.

With θ made zero, (10) properly reduced, assumes the special form:

$$
(12)\qquad (1-x)^m \sum_{t=0}^{t=n} c_{m-1+t,\, m-1} x^t = 1 - \sum_{t=0}^{t=m-1} c_{n+t,\, t} (1-x)^t x^{n+1},
$$

which can be viewed in three different aspects.

Written in the form:

$$
\sum_{t=0}^{t=n} c_{m-1+t,\, m-1} x^t = \frac{1}{(1-x)^m} - \frac{1}{(1-x)^m} \sum_{t=0}^{t=m-1} c_{n+t,\, t} (1-x)^t x^{n+1},
$$

it represents in its second member, in a definite number of terms, the sum of the series in the first member, consisting of any number of terms. The first expression of the second member being invariant as to n, becomes the sum of the series when the second, dependent upon n, vanishes by special values of n or x. Thus, for $m = 2$, this second expression becomes zero when $x = (n+2)/(n+1)$; and this, substituted in (12), gives the identity:

$$
(13)\qquad 1 + 2\left(\frac{n+1}{n}\right) + 3\left(\frac{n+1}{n}\right)^2 + \cdots + n\left(\frac{n+1}{n}\right)^{n-1} = n^2.
$$

If the second expression in the previous formula becomes indefinitely small as n indefinitely increases, we have the summation of the infinite convergent series.

Written in the form:

$$
\frac{1}{(1-x)^m} = \sum_{t=0}^{t=n} c_{m-1+t,\, m-1} x^t + \frac{1}{(1-x)^m} \sum_{t=0}^{t=m-1} c_{n+t,\, t} (1-x)^t x^{n+1},
$$

it represents the development of the binomial with negative exponents, *i.e.*, $(1-x)^{-m}$, into a finite series with a remainder term. The development is therefore good whether the series is finite, convergent or divergent.

If considered in the order:

$$\frac{1-\sum_{t=0}^{t=m-1} c_{n+t,\,t}(1-x)^t x^{n+1}}{(1-x)^m} = \sum_{t=0}^{t=n} c_{m-1+t,\,m-1}x^t,$$

it presents an extension of the theorem, that the difference of the same powers is divisible by the difference of the first powers.

Let, for illustration, $n = 3$, $m = 1, 2, 3, \cdots$, and we shall have successively:

$$\frac{1-x^4}{1-x} = 1 + x + x^2 + x^3,$$

$$\frac{1-x^4-4(1-x)x^4}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3,$$

$$\frac{1-x^4-4(1-x)x^4-10(1-x)^2x^4}{(1-x)^3} = 1 + 3x + 6x^2 + 10x^3,$$

etc.; where the ubiquitous binomial coefficients are recognized in their usual rôle.

IV. AN ERRONEOUS RULE FOR FINDING A HYPOTENUSE, WITH A COROLLARY.

By M. W. JACOBS, JR., Harrisburg, Pa.

To find the hypotenuse of a right-angled triangle, take half the length of the longer leg and add the length of the shorter leg. This was the rule nearly all the boys of an arithmetic class were found to be using. Much to the instructor's surprise, the correct answer was thus obtained to almost all of the examples. An inspection of the problems correctly solved in this manner showed that the ratio of the longer to the shorter leg was 4 : 3.

That this is the only ratio for which the boys' rule holds true may be demonstrated as follows: Let a = longer leg; b = shorter leg; h = hypotenuse. By the boys' rule

$$h = \frac{1}{2}a + b = \frac{a+2b}{2}.$$

Then

$$h^2 = \frac{a^2+4ab+4b^2}{4}.$$

But $h^2 = a^2 + b^2$. Hence

$$a^2 + b^2 = \frac{a^2+4ab+4b^2}{4}.$$

Then $4a^2 + 4b^2 = a^2 + 4ab + 4b^2$; whence $3a^2 = 4ab$; therefore $a/b = 4/3$. The